

Fermions in energon field

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The behavior of fermions in the gauge field created by the energon, a recently found classical solution of the non-Abelian gauge theory, is considered. The spectrum of fermions is evaluated explicitly for the case when parameters governing the energon are chosen to make it look similar to the chain of separated instantons and antiinstantons. The zero eigenvalue in the spectrum of the Dirac operator is found. The role of a specific discrete symmetry, which distinguishes the energon, is discussed.

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I. INTRODUCTION

The semiclassical approach to quantum Yang-Mills field is known to describe a number of its important properties. This approach is based on classical solutions. An important class of these solutions is the self-dual solutions. The firstly discovered BPST instanton of Ref.[1] remains one of the most useful tools, for a review see e.g. Ref.[2]. The general multi-instanton solution is described by the ADHM construction of Ref.[3].

A modification of the instanton solution, which is known as the caloron of Refs. [4, 5], has a periodic nature that makes it interesting for applications at finite temperatures. Recent developments and further references for the caloron can be found in reviews Refs. [6, 7]. The periodic conditions were differently implemented in the so called periodic instantons of Refs. [8]; the configuration proved useful in multiparticle production and studies of the baryon and lepton number violation.

Ref.[9] introduced the sphaleron, a self-consistent static solution for interacting gauge and Higgs fields. It was argued in Ref. [10] that the pure gauge field can exhibit the sphaleron-type behavior when it is treated using particular restrictions. The found gauge field configuration, called COS-sphaleron, possesses only the magnetic field.

Another line for generalization of classical solutions was pursued in Refs. [11, 12]. The idea was to take a minimum of the classical action under the condition that the Euclidean energy is fixed; the corresponding field configuration was called the energon. Hence the energy plays the role of an additional parameter, which can be tuned to adjust properties of the energon to the needs of a problem at hand. In [11] it was shown that for finite temperatures the energon provides exponential enhancement for the probability of tunneling through the potential barrier separating regions with different topological charges.

A semiclassical solution in gauge theory usually incorporates strong gauge fields. It is important therefore to

consider their influence on fermions, which may have interesting consequences. For example, for instantons the fermionic zero modes have valuable physical manifestations [13, 14]; they also give an interesting example of the application of the Atiyah-Singer theorem [15].

The present work considers the behavior of fermions in the field of the energon. In order to simplify the problem the parameters governing the energon configuration are tuned in such a way that the gauge field of the energon looks similar to the field created by a periodic chain of well separated instantons and antiinstantons. The small parameter $\beta = \rho/T \ll 1$, where ρ is the radius of the mentioned instantons and antiinstantons and T is their separation, is used to solve the problem analytically. A continuum spectrum of eigenstates of the Dirac operator is found explicitly.

The corresponding eigenvalues of the Dirac operator are small, which agrees with the fact that the field of the energon chosen resembles a field of well separated instantons and antiinstantons. However, generally these eigenvalues differ from zero. Interestingly, there exist two special states, which are annihilated by the Dirac operator. In other words, the eigenvalue of the Dirac operator for these two states turns zero, and hence the energon possesses the fermionic zero modes.

II. FERMIONS IN ENERAGON FIELD

We presume that the gauge group is $SU(2)$, fermions belong to its fundamental representation; Euclidean formulation of the theory is chosen. Consider the Dirac equation for massless fermions

$$i \gamma_\mu \nabla_\mu \psi = \varepsilon \psi. \quad (2.1)$$

Here ε and ψ play the roles of the eigenvalue and corresponding eigenfunction of the Dirac operator, which in the Euclidean formulation is Hermitian. In this Section we will formulate the eigenproblem for Eq.(2.1) when the potential A_μ^a , which appears in the covariant derivative $\nabla_\mu = \partial_\mu - i A_\mu^a \tau^a / 2$, is created by the energon.

To clarify notation note that we use the Euclidean

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Dirac matrices γ_μ defined as follows

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu^+ \\ \sigma_\mu^- & 0 \end{pmatrix}, \quad (2.2)$$

where $\mu = 1, \dots, 4$ and the Pauli matrices read

$$\sigma_\mu^\pm = (\pm i \boldsymbol{\sigma}, 1). \quad (2.3)$$

We need first to outline basic properties of the energon solution of [11, 12]. The energon provides a conditional minimum for the Euclidean action achieved when the Euclidean energy is fixed. The classical action S and Euclidean energy \mathcal{E} of the SU(2) gauge field read

$$S = \int (K + V) d\tau, \quad \mathcal{E} = K - V. \quad (2.4)$$

Here

$$K = \frac{1}{2g^2} \int \mathbf{E}^a \cdot \mathbf{E}^a d^3r, \quad V = \frac{1}{2g^2} \int \mathbf{B}^a \cdot \mathbf{B}^a d^3r \quad (2.5)$$

are expressed via the electric $E_m^a = F_{m4}^a$ and magnetic $B_m^a = \frac{1}{2}\epsilon_{mnl}F_{nl}^a$ components of the gauge field $F_{\mu\nu}^a$ ($a = 1, 2, 3$ is an isotopic index; $m, n, l = 1, 2, 3$ are indexes of the 3D coordinate space, whose presence is replaced in (2.5) by bold and dot-product notation). The signs chosen in front of V in Eqs.(2.4) comply with the Euclidean description.

As was mentioned, the energon provides the minimum for the action and hence satisfies

$$\delta S = 0. \quad (2.6)$$

The minimum is considered under the restriction that \mathcal{E} takes a chosen, fixed value. Hence \mathcal{E} can be considered as an additional parameter.

Equations (2.4)-(2.6) have a surprisingly simple solution [11, 12]. One considers firstly a non-restricted solution of (2.6), which can be chosen as any self-dual, or anti-selfdual solution $A_{\mu,SD}^a(x)$. Secondly, alongside conventional Euclidean coordinates $x_\mu = (\mathbf{r}, \tau)$ one introduces also the modified coordinates

$$z_\mu = (\mathbf{r}, q(\tau)), \quad (2.7)$$

in which the Euclidean time τ is replaced by a function $q(\tau)$ specified below, see Eq.(2.9).

We can write down the energon solution of Eqs. (2.4)-(2.6) as follows

$$A_m^a(x) = A_{m,SD}^a(z), \quad A_4^a(x) = A_{4,SD}^a(z) \dot{q}(\tau), \quad (2.8)$$

where $\dot{q}(\tau) = \frac{dq}{d\tau}(\tau)$. Note the difference in coordinates in the left and right hand sides here. The function $q(\tau)$ should be found from the condition specifying the energy, which reads

$$\mathcal{E} = (\dot{q}^2 - 1) \frac{1}{2g^2} \int \mathbf{B}_{SD}^a(z) \cdot \mathbf{B}_{SD}^a(z) d^3r. \quad (2.9)$$

The integral in the right-hand side here is a function of q , which is defined by the chosen self-dual solution. Consequently, Eq.(2.9) can be considered an ordinary differential equation on $q(\tau)$. Its solution finalizes description of the energon potential in Eq.(2.8).

Thus for each self-dual solution $A_{\mu,SD}^a$ there exists a set of energon solutions A_μ^a parametrized by a value of \mathcal{E} . In particular, for $\mathcal{E} = 0$ the energon is reduced to the underlying self-dual solution, as can be seen from Eq.(2.9), which in this case gives $q(\tau) = \tau$; consequently one deduces that $x_\mu = z_\mu$, and $A_\mu(x) = A_{\mu,SD}(x)$.

We will restrict further discussion to the case of negative \mathcal{E} . Equation (2.9) ensures in this case that $\dot{q}^2 < 1$. Assuming that the self-dual solution is well localized in 4D Euclidean space one observes also that the integral in Eq.(2.9) decreases at large q . These two conditions together indicate that $q(\tau)$ is an oscillating function of τ . The variation of τ over the period T results in the variation of $q(\tau)$ between its minimum and maximum, where $\dot{q} = 0$. An explicit expression for the period can be written in the form

$$T = \oint \frac{dq}{\dot{q}}, \quad (2.10)$$

where \dot{q} should be taken from (2.9).

Thus, under conditions specified above the energon potential in Eq.(2.8) and corresponding field $F_{\mu\nu}^a$ are periodic functions of τ with the period T . At the same time $F_{\mu\nu}^a$ is well localized as a function of 3D coordinates \mathbf{r} , $F_{\mu\nu}^a \rightarrow 0$ when $\mathbf{r} \rightarrow \infty$.

Returning to the eigenvalue problem in Eq.(2.1) we can now be more specific about properties of ψ . Firstly, the localization of the energon in \mathbf{r} makes it possible, and obviously appealing, to restrict our consideration to those $\psi = \psi(\mathbf{r}, \tau)$, which are localized functions of \mathbf{r} ,

$$\psi(\mathbf{r}, \tau) \rightarrow 0, \quad \mathbf{r} \rightarrow \infty. \quad (2.11)$$

Secondly, we need to take into account that being periodic in τ , the energon potential makes the Dirac operator in Eq.(2.1) a periodic operator as well. Consequently, we can impose on ψ the following condition

$$\psi(\mathbf{r}, \tau + T) = e^{i\omega T} \psi(\mathbf{r}, \tau). \quad (2.12)$$

It represents for the case at hand Bloch's theorem, which is commonly used in description of crystals in condense matter, see e.g. Ref. [16]. Equation (2.12) makes it clear that the parameter ω , which specifies properties of the solution, can be chosen to satisfy the following condition

$$-1 \leq \omega T / \pi \leq 1. \quad (2.13)$$

This region of ωT can be called the first Brillouin zone having in mind similar conditions specifying the zone structure in crystals [16].

Summarizing, Eqs. (2.1), (2.11) and (2.12) formulate the eigenvalue problem for massless fermions in the energon field defined by Eqs. (2.7), (2.8) and (2.9). According to Bloch's theorem (2.12) the eigenfunction and

eigenvalue depend on the parameter ω , which belongs to the first Brillouin zone (2.13). The ω -dependence will be marked by a subscript, $\psi = \psi_\omega$ and $\varepsilon = \varepsilon_\omega$.

III. DISCRETE R-SYMMETRY

Consider an important discrete symmetry that exists in the problem. Its presence can be described in general terms. However, in order to simplify presentation we restrict our discussion to the energon, which is constructed from the BPST instanton. The instanton potential reads

$$A_{\mu,\text{in}}^a(x) = \bar{\eta}_{\mu\nu}^a \frac{x_\nu \tau^a}{x^2(x^2 + 1)}. \quad (3.1)$$

Here $\bar{\eta}_{\mu\nu}^a$ are the t'Hooft symbols, t'Hooft singular gauge is presumed, the instanton center is chosen at the origin, and units are scaled to make the instanton radius unity. It is useful to present the instanton potential from (3.1) in a more detailed 3D notation

$$\mathbf{A}_{\text{in}}(\mathbf{r}, \tau) = \frac{\mathbf{r} \times \boldsymbol{\tau} - \tau \boldsymbol{\tau}}{x^2(x^2 + 1)}, \quad A_{4,\text{in}}(\mathbf{r}, \tau) = \frac{\mathbf{r} \cdot \boldsymbol{\tau}}{x^2(x^2 + 1)}. \quad (3.2)$$

To avoid confusion, τ is the Euclidean time and $x^2 = r^2 + \tau^2$, whereas the bold $\boldsymbol{\tau}$ represents a triplet of the Pauli matrices. The antiinstanton solution can also be described by (3.1), provided $\bar{\eta}_{\mu\nu}^a$ is replaced there by $\eta_{\mu\nu}^a$. The known consequence of this fact is that the antiinstanton potential can be obtained from the instanton one by applying either the operator \hat{P} of the inversion of coordinates \mathbf{r} , or the operator \hat{T} of the reversal of time τ . As a result the instanton potential remains invariant under a combined influence of $\hat{P}\hat{T}$,

$$\hat{P}\hat{T} A_{\mu,\text{in}}^a = A_{\mu,\text{in}}^a. \quad (3.3)$$

Let us show that there exists a similar discrete symmetry for the energon solution. Following the procedure outlined in Section II we can build the energon potential starting from the instanton solution (3.2) as follows

$$\mathbf{A}(\mathbf{r}, \tau) = \frac{\mathbf{r} \times \boldsymbol{\tau} - q(\tau) \boldsymbol{\tau}}{z^2(z^2 + 1)}, \quad A_4(\mathbf{r}, \tau) = \frac{\dot{q}(\tau) (\mathbf{r} \cdot \boldsymbol{\tau})}{z^2(z^2 + 1)}. \quad (3.4)$$

Here $z^2 = r^2 + q^2(\tau)$. To fully describe the energon one needs to resolve Eq.(2.9) on $q(\tau)$. An additive constant arising during this procedure can be chosen in such a way as to satisfy $q(0) = 0$. After that it is easy to see that the periodic function $q(\tau)$ changes the sign over the half-period

$$q(\tau + T/2) = -q(\tau), \quad \dot{q}(\tau + T/2) = -\dot{q}(\tau). \quad (3.5)$$

Consider now the following operator

$$\hat{R} = \hat{P} e^{i(T/2) \partial_\tau}. \quad (3.6)$$

Here $\exp(i(T/2) \partial_\tau)$ is the operator of a shift of the argument τ of any function $\phi(\tau)$ by the half-period of the

energon, $\exp(i(T/2) \partial_\tau) \phi(\tau) = \phi(\tau + T/2)$. Remembering Eqs.(3.5) one finds that the energon potential defined in (3.4) is R -invariant

$$\hat{R} A_\mu^a(x) = A_\mu^a(x). \quad (3.7)$$

Observe a similarity with Eq.(3.3). The difference is that instead of the time inversion, which appears in (3.3) for the instanton case, Eq.(3.7) incorporates the shift of τ by $T/2$.

The invariance of the energon potential under \hat{R} makes the Dirac operator invariant as well

$$\hat{R} i \gamma_\mu \nabla_\mu \psi = i \gamma_\mu \nabla_\mu \hat{R} \psi. \quad (3.8)$$

As a result the eigenstate ψ of the Dirac operator should also be an eigenstate of \hat{R} , $\hat{R} \psi = \lambda \psi$, where λ is the eigenvalue. The definition of \hat{R} (3.6) implies $\hat{R}^2 = \exp(i T \partial_\tau)$, where the resulting operator describes a shift of τ by the energon period T . Remember that this operator is also present in Bloch's periodic conditions (2.12) which can be written as $\exp(i T \partial_\tau) \psi(\tau) = \exp(i \omega T) \psi(\tau + T)$. We derive from this fact that λ can take only the following values, $\lambda = \pm \exp(i \omega T/2)$.

Thus, the function ψ can be characterized by two parameters, one is ω , another is the sign, which distinguishes two different values of λ for a given ω . It makes sense therefore to mark the fermion function by these parameters, $\psi = \psi_{\omega,\pm}$ presuming that its transformation under \hat{R} reads

$$\hat{R} \psi_{\omega,\pm} = \pm \exp(i \omega T/2) \psi_{\omega,\pm}. \quad (3.9)$$

Observe that Bloch's periodic condition (2.12) can be derived from (3.9) because, as was mentioned, $\hat{R}^2 = \exp(i T \partial_\tau)$. We will see below that (3.9) has a strong impact on the fermion spectrum.

IV. LOW EUCLIDEAN ENERGIES

Consider the case where Euclidean energy is negative and small $\mathcal{E} < 0$, $a|\mathcal{E}| \ll 1$, where a is a typical distance at which the gauge field varies substantially. Equation (2.10) shows that the period T in this case is large. To be specific we take the energon, whose vector potential is written in (3.4). It was found in [11] that the relation between the energon period T and small Euclidean energy \mathcal{E} in this case reads

$$\frac{T}{2\rho} \approx 2.51 \left(-\frac{3\pi^2}{g^2 \rho \mathcal{E}} \right)^{1/5}. \quad (4.1)$$

Here g is the coupling constant of the SU(2) gauge theory. Previously we chose the instanton radius as unity, $\rho = 1$, but here present it explicitly to make formulas more transparent. For small energies, $\rho|\mathcal{E}| \ll 1$, Eq.(4.1) implies

$$\beta = \rho/T \ll 1. \quad (4.2)$$

Observe that for small β the gauge field produced by the energon can be strong only if τ satisfies either condition $|\tau - sT| \lesssim \rho$, or $|\tau - (s + \frac{1}{2})T| \lesssim \rho$ where $s = 0, \pm 1 \dots$. In the first case, $|\tau - sT| \lesssim \rho$, the field created by the energon is close to the field of the instanton located at $x_\mu = (0, 0, 0, sT)$. In the second case, $|\tau - (s + \frac{1}{2})T| \lesssim \rho$, the energon field is close to the field created by the antiinstanton located at $x_\mu = (0, 0, 0, (s + 1/2)T)$.

Correspondingly, at small negative \mathcal{E} the energon field can be approximated by the field of a chain of alternating instantons and antiinstanton, which is stretched along the τ -axis being also periodic in τ . The separation between the closest pair of instantons or antiinstantons equals the energon period T , while the nearest instanton and antiinstanton are separated by $T/2$. The radius, as well as the orientation of all instantons and antiinstantons in this chain are the same.

Remember that the pure instanton and pure antiinstanton configurations possess fermionic zero modes. This implies that any field configuration that includes well separated instantons and antiinstantons should possess fermion modes, which satisfy the Dirac equation (2.1) with small eigenvalues ε , $|\varepsilon|\rho \ll 1$.

Let us find these eigenvalues and the corresponding eigenfunctions ψ for the energon (3.4), assuming that Eq.(4.2) is valid. We can apply the method of “strong coupling”, which is similar to the one used conventionally in solid state physics, see e.g. Section 10 in [16]. It is commonly applied for constructing the wave functions of electrons propagating in crystals from the wave functions of atomic electrons, which are localized on individual atoms. The smaller the overlapping integrals between the wave functions localized on different atoms, the better the method works.

In the case at hand, instead of a crystal field we have the gauge field of the energon periodic in τ . The role of atoms is played by instantons and antiinstantons. Instead of atomic wave functions localized on individual atoms we have fermionic zero-modes localized on separate instantons and antiinstantons. The non-relativistic Schrödinger equation is replaced by the Dirac equation. However, one can see that these distinctions do not play an essential role. The decisive point is that in order to make the strong coupling method applicable in our case, the fermionic wave function should be large only within well separated regions, where the gauge field is large. We will see below that this is really the case.

The zero fermion mode, which exists in the field of the instanton (3.1) localized at the origin, reads

$$\psi_{\text{in}}^{(0)}(x) = \frac{1}{\pi} \frac{n_\mu}{(x^2 + 1)^{3/2}} \begin{pmatrix} \sigma_\mu^+ \epsilon \\ 0 \end{pmatrix}. \quad (4.3)$$

Here $n_\mu = x_\mu/\sqrt{x^2}$. The fermion wave function $\psi_{\text{in}}^{(0)}(x)$ is written here as a chiral Dirac 4-spinor and isotopic doublet. It possesses the index $m = 1, 2$ (not written explicitly in (4.3)), which distinguishes the two nonzero components in the Dirac spinor. It also possesses the

index $\alpha = 1, 2$ (also not shown in (4.3) explicitly), which describes the isotopic doublet. These two indexes are blended together in the matrix $\sigma_\mu^+ \epsilon \equiv (\sigma_\mu^+ \epsilon)_{m,\alpha}$, where ϵ is the 2×2 antisymmetric matrix, $\epsilon_{12} = -\epsilon_{21} = 1$.

We also need the fermion zero mode of the antiinstanton. Presuming temporally that the antiinstanton is located at the origin we can write this mode as follows

$$\psi_{\text{a-in}}^{(0)} = \frac{1}{\pi} \frac{n_\mu}{(x^2 + 1)^{3/2}} \begin{pmatrix} 0 \\ \sigma_\mu^- \epsilon \end{pmatrix}. \quad (4.4)$$

Following the spirit of the strong coupling approach [16] we take a linear combination of the instanton zero modes localized at all those points, where the energon potential is close to the potential created by the instanton

$$\Psi_{\omega,\text{in}}(\tau) = \frac{1}{\sqrt{N}} \sum_s e^{is\omega T} \psi_{\text{in}}^{(0)}(\tau - sT). \quad (4.5)$$

The coefficients $\exp(is\omega T)$ in this linear combination are completely defined by Bloch’s conditions (2.12), N is a large integer, which in the final stage of calculations will be taken to the limit $N \rightarrow \infty$. Summation over an integer s covers the region $-N/2 < s < N/2$. The parameter ω in the intermediate calculations should satisfy $\omega = 2\pi n/N$, where n is an integer, but in the limit $N, n \rightarrow \infty$ this parameter can take any value inside the Brillouin zone (2.13). For simplicity of presentation the argument \mathbf{r} of the zero mode in (4.5) is suppressed. One immediately verifies that $\Psi_{\omega,\text{in}}(\tau)$ satisfies Bloch’s periodic condition (2.12).

Similarly, we construct the fermion wave function using the antiinstanton zero modes

$$\Psi_{\omega,\text{a-in}}(\tau) = \frac{1}{\sqrt{N}} \sum_s e^{i(s+\frac{1}{2})\omega T} \psi_{\text{a-in}}^{(0)}(\tau - (s + \frac{1}{2})T). \quad (4.6)$$

The argument $\tau - (s + \frac{1}{2})T$ of the wave function of fermionic zero modes takes into account the fact that the energon potential is close to the potential of the antiinstanton in the vicinities of each point $\tau = (s + \frac{1}{2})T$. This fact makes it also convenient to present s in the argument of the exponent function in (4.6) as $(s + \frac{1}{2})\omega T$. One verifies that the function $\Psi_{\omega,\text{a-in}}(\tau)$ complies with Bloch’s condition.

We have now to account for the discrete R-symmetry, which is present in the energon potential. The commutation of the Dirac operator with the operator \hat{R} stated in (3.8) implies that the functions used to diagonalize the Dirac operator should also be eigenfunctions of the operator \hat{R} . Meanwhile the functions $\Psi_{\omega,\text{in}}(\tau)$ and $\Psi_{\omega,\text{a-in}}(\tau)$ are not the eigenfunctions of \hat{R} because their transformations read

$$\hat{R} \Psi_{\omega,\text{in}} = e^{i\omega T/2} \Psi_{\omega,\text{a-in}}(x), \quad (4.7)$$

$$\hat{R} \Psi_{\omega,\text{a-in}} = e^{i\omega T/2} \Psi_{\omega,\text{in}}(x). \quad (4.8)$$

To remedy this deficiency take the linear combinations

$$\Psi_{\omega,\pm} = \frac{1}{\sqrt{2}} (\Psi_{\omega,\text{in}} \pm \Psi_{\omega,\text{a-in}}). \quad (4.9)$$

Using (4.7) and (4.8) one finds for these functions

$$\hat{R}\Psi_{\omega,\pm} = \pm e^{i\omega T/2}\Psi_{\omega,\pm}. \quad (4.10)$$

Deriving (4.10) one takes into account that the operator of inversion changes the chirality of 4-spinors as illustrated by the identity [23]

$$\hat{P}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.11)$$

Equation (4.10) states that $\Psi_{\omega,\pm}$ are eigenfunctions of \hat{R} , precisely as required by the symmetry condition (3.9). The discussion presented after (3.9) implies that Eq.(4.10) automatically ensures the validity of Bloch's conditions (2.12).

One can see that there is only one possible way for constructing a fermionic wave function, which is a linear combination of the fermion zero modes related to instantons and antiinstantons, and which satisfies the necessary symmetry conditions. The function in question $\Psi_{\omega,\sigma}$ from (4.9) possesses two parameters, ω and $\sigma = \pm 1$. In other words, for a chosen parameters ω, σ the function is unique. As a result the strong coupling method greatly simplifies. The expression for the eigenvalue in this case simply reads [24]

$$\varepsilon_{\omega,\pm} = \frac{\langle \Psi_{\omega,\pm} | i\gamma_\mu \nabla_\mu | \Psi_{\omega,\pm} \rangle}{\langle \Psi_{\omega,\pm} | \Psi_{\omega,\pm} \rangle} \equiv \frac{\mathcal{H}}{\mathcal{N}}. \quad (4.12)$$

Here \mathcal{H} and \mathcal{N} equal the nominator and denominator of the expression in the middle. Notation in (4.12) presumes that matrix elements are calculated as integrals over the 4D Euclidean space.

Straightforward calculations, see (A2) in Appendix A, show that

$$\mathcal{N} = \langle \Psi_{\omega,\pm} | \Psi_{\omega,\pm} \rangle = 1 + O(\beta^2). \quad (4.13)$$

Hence, hunting for the main term in an expansion of $\varepsilon_{\omega,\pm}$ over β , we can restrict ourselves to a trivial approximation $\mathcal{N} \approx 1$. This complies with conditions of applicability of the strong coupling approximation, which requires the overlapping integrals between functions located at different centers be small.

Thus, in order to find the eigenvalues $\varepsilon_{\omega,\pm}$ we need only to calculate the matrix element \mathcal{H} . First of all, using the fact that γ_μ matrices mix chirality we can rewrite it as follows

$$\mathcal{H} = \pm \text{Re} \langle \Psi_{\omega,\text{in}} | i\gamma_\mu \nabla_\mu | \Psi_{\omega,\text{a-in}} \rangle, \quad (4.14)$$

where the sign in front complies with the sign in the chosen function $\Psi_{\omega,\pm}$. Secondly, using Eqs.(4.5) and (4.6), we rewrite this expression once again

$$\langle \Psi_{\omega,\text{in}} | i\gamma_\mu \nabla_\mu | \Psi_{\omega,\text{a-in}} \rangle = \sum_s e^{i(s+\frac{1}{2})\omega T} h_s, \quad (4.15)$$

where

$$h_s = \langle \psi_{\text{in}}^{(0)}(\tau) | i\gamma_\mu \nabla_\mu | \psi_{\text{a-in}}^{(0)}(\tau - (s+\frac{1}{2})T) \rangle. \quad (4.16)$$

In Eq.(4.15) we take the desired limit $N \rightarrow \infty$ and presume accordingly that $-\infty < s < \infty$.

The main contribution to the integral over τ in (4.16) comes from the region located between the origin, $\tau = 0$, where the function $\psi_{\text{in}}^{(0)}(\mathbf{r}, \tau)$ is large and the point $\tau = (s + \frac{1}{2})T$, where the function $\psi_{\text{a-in}}^{(0)}(\mathbf{r}, \tau - (s + \frac{1}{2})T)$ is prominent. Moreover, it is shown in Appendix B that the integral over τ collapses to the intermediate point $\tau = \tau_0 = \frac{1}{2}(s + \frac{1}{2})T$. This is stated in Eq.(B3), which we reiterate here

$$h_s = i \text{sign}(s + \frac{1}{2}) \int \psi_{\text{in}}^{(0)+}(\tau_0) \gamma_4 \psi_{\text{a-in}}^{(0)}(-\tau_0) d^3r. \quad (4.17)$$

Observe that both functions $\psi_{\text{in}}^{(0)}$ and $\psi_{\text{a-in}}^{(0)}$ appear in (4.17) only for large values of the variable τ , $\tau = \pm\tau_0$, where the potential created by the energon is small because $|\tau_0| \gg \rho$. Correspondingly, in this region the asymptotic forms of the equations for the fermion wave function are satisfied by both the instanton and anti-instanton modes, $\gamma_\mu \partial_\mu \psi_{\text{in}}^{(0)} \approx \gamma_\mu \partial_\mu \psi_{\text{a-in}}^{(0)} \approx 0$. This shows that the wave functions in (4.17) are used only within the region where their validity is justified. Equation (4.17) can be compared with the known Holstein-Herring method, also called Smirnov's method, which is commonly used in molecular and condense matter calculations [18–20], for a brief description see also chapters 50 and 81 of [22].

Direct calculations in (4.17), see (B7) in Appendix B, give

$$h_s \approx -\frac{4i\beta^3}{(s + \frac{1}{2})^3}. \quad (4.18)$$

One substitutes now (4.18) into (4.15) and rearranges the summation over s there to make it running over $s \geq 0$. After that Eqs.(4.12), (4.13) and (4.14) give

$$\varepsilon_{\omega,\pm} \approx \pm 8\beta^3 \sum_{s=0}^{\infty} \frac{\sin\left((s + \frac{1}{2})\omega T\right)}{(s + \frac{1}{2})^3}. \quad (4.19)$$

Using equation 5.4.6.13 from Ref. [21] we finally derive a simple, appealing result

$$\varepsilon_{\omega,\pm} = \pm 4\pi^2\beta^3\omega T \left(1 - \frac{|\omega|T}{2\pi}\right). \quad (4.20)$$

Here ωT belongs to the first Brillouin zone (2.13) and $\beta = \rho/T$ is the small gas parameter. Equation (4.20) is one of the important results of this work. Its discussion is given in Section V.

V. DISCUSSION

Consider properties of the eigenvalues $\varepsilon_{\omega,\pm}$ of the Dirac operator presented in Eq.(4.20). Our initial idea was that conditions $\mathcal{E} < 0$, $|\mathcal{E}|\rho \ll 1$ should bring into existence fermion modes with properties similar to fermion

zero modes of instantons and antiinstantons. Equation (4.20) complies with these expectations. The eigenvalues found there are small, $\propto \beta^3$ [25]. The smallness of the eigenvalues also justifies the validity of the strong coupling approach, which was implemented in Eqs.(4.9). The eigenfunctions (4.9) are localized in 3D coordinate space, $\Psi_{\omega,\pm}(\mathbf{r}, \tau) \rightarrow 0$ when $\mathbf{r} \rightarrow \infty$, in agreement with Eq.(2.11). This property is similar to the behavior of the instanton zero modes.

However, there is a notable distinction between $\Psi_{\omega,\pm}(\mathbf{r}, \tau)$ and conventional fermionic zero modes, which appear for self-dual gauge fields. The latter decrease for large τ , while the former satisfy Bloch's periodic condition (2.12), which originates from the periodic nature of the gauge field created by the energon.

It is well known, see e.g. [16], that on the boundaries of the Brillouin zone the derivative of the eigenvalue should vanish. In our case this condition reads

$$\frac{\partial \varepsilon_{\omega,\pm}}{\partial \omega} = 0, \quad \frac{\omega T}{\pi} = \pm 1. \quad (5.1)$$

It is gratifying that Eq.(4.20) complies with this necessity.

Notably, the spectrum of eigenvalues of the Dirac operator possesses the doubly degenerate zero eigenvalue, $\varepsilon_{0,\pm} = 0$, which is present at the center of the Brillouin zone $\omega = 0$. Equation (4.10) for $\omega = 0$ shows that the two corresponding fermion states are transformed under the \hat{R} operator as follows

$$\hat{R} \Psi_{0,\pm} = \pm \Psi_{0,\pm}, \quad (5.2)$$

It is tempting to conjecture that the very presence of states with zero eigenvalues of the Dirac operator in the fermion spectrum can be attributed to the existence of the R-symmetry (3.7) for the energon solution. It would be interesting to justify this point on the basis of arguments of a general nature, appealing probably to the Atiya-Singer theorem. However, we will not make such attempt in the present work since we restrict our techniques here to the simple strong coupling approach. (The importance of the R-symmetry can be probed in this technique by applying the method of strong coupling to a set of functions, which do not satisfy the symmetry conditions (3.9), for example taking a pair of functions $\Psi_{\omega,\text{in}}, \Psi_{\omega,\text{a-in}}$ from (4.5),(4.6). One verifies that in this case the spectrum of eigenvalues of the Dirac operator has no zero eigenvalues. This simple argument can be considered as a hint, more accurate consideration is needed.)

The results reported warrant further investigation. It would be interesting to consider properties of the fermion spectrum in the energon field in general terms, possibly proving the presence of zero eigenvalues of the Dirac operator for an arbitrary energon configuration. The periodic in τ nature of the energon field makes such studies an appealing task by itself, whereas an interplay of ideas with solid state physics provides additional impetus. The existence of zero eigenvalues of the Dirac operator inspires consideration of creation of fermion-antifermion pairs in

the field of the energon, similar to pair creation by instantons.

Appendix A: Normalization

Consider the normalization factor \mathcal{N} , which arises in the denominator of Eq.(4.12). Using Eq.(4.9) and then Eqs.(4.5), (4.6) together with (4.3) and (4.4) we present it as follows

$$\begin{aligned} \mathcal{N} &= \langle \Psi_{\omega,\pm} | \Psi_{\omega,\pm} \rangle = \langle \Psi_{\omega,\text{in}} | \Psi_{\omega,\text{in}} \rangle = \\ &= \sum_{s=-\infty}^{\infty} e^{i s \omega T} \langle \psi_{\text{in}}^{(0)}(\tau) | \psi_{\text{in}}^{(0)}(\tau - sT) \rangle = \\ &= \sum_{s=-\infty}^{\infty} e^{i s \omega T} \frac{2}{\pi^2} \int \frac{n_{\mu} n'_{\mu}}{(x^2 + 1)^{3/2} (x'^2 + 1)^{3/2}} d^4 x \quad (A1) \end{aligned}$$

Here $x = (\mathbf{r}, \tau)$, $n_{\mu} = x_{\mu}/\sqrt{x^2}$, and $x'_{\mu} = (\mathbf{r}, \tau - sT)$, $n'_{\mu} = x'_{\mu}/\sqrt{x'^2}$. Straightforward calculation of the integral in (A1) and obvious rearrangements in the summation (to make it running over positive s only) gives

$$\begin{aligned} \mathcal{N} &\approx 1 + 4\beta^2 \sum_{s=1}^{\infty} \frac{\cos(s\omega T)}{s^2} = \\ &= 1 + 4\pi^2 \beta^2 \left(\left(\frac{\omega T}{2\pi} \right)^2 - \frac{|\omega|T}{2\pi} + \frac{1}{6} \right). \quad (A2) \end{aligned}$$

The last identity here is written using equation 5.4.2.7 from [21].

Appendix B: Matrix element

Let us simplify the matrix element \mathcal{H} , which appears in the nominator of (4.12), by calculating h_s from Eqs.(4.15). Presuming firstly that $s \geq 0$ let us rewrite (4.16) as follows

$$\begin{aligned} h_s &= \int d^3 r \left(\int_{-\infty}^{\tau_0} d\tau + \int_{\tau_0}^{\infty} d\tau \right) \times \\ &\quad \left(\psi_{\text{in}}^{(0)}(\mathbf{r}, \tau) \right)^+ i \gamma_{\mu} \nabla_{\mu} \psi_{\text{a-in}}^{(0)} \left(\mathbf{r}, \tau - \left(s + \frac{1}{2} \right) T \right). \quad (B1) \end{aligned}$$

Here an obvious bridging notation is used for integrals over τ . It is convenient to choose the intermediate integration limit τ_0 strictly between the origin and the point $(s + \frac{1}{2})T$,

$$\tau_0 = \frac{1}{2} \left(s + \frac{1}{2} \right) T. \quad (B2)$$

The idea behind Eqs.(B1), (B2) is that the integral over τ is saturated in the vicinity of τ_0 . Similar matrix elements have long being used in molecular and solid state physics [18–20]. For the region $-T/4 < \tau < \tau_0$ the potential created by the energon is either small, or (if large) is close

to the potential created by the instanton located at the origin. The first option takes place when $r^2 + \tau^2 > 1$, the second when $r^2 + \tau^2 \leq 1$. In both cases we can approximate the potential created by the energon by the instanton potential and consequently write $\nabla_\mu \approx \nabla_{\mu,\text{in}}$. Meanwhile, the region $\tau < -T/4$ is of no importance since both wave functions from (B1) are small there. Thus, an approximation $\nabla_\mu \approx \nabla_{\mu,\text{in}}$ is applicable for all $\tau < \tau_0$. Similarly, in the region $\tau_0 < \tau$ we can approximate the energon potential by the potential created by the antiinstanton, which is located at $\mathbf{r}_{\text{a-in}} = 0$ and $\tau_{\text{a-in}} = T/2$. In this region one writes therefore $\nabla_\mu \approx \nabla_{\mu,\text{a-in}}$. This last relation implies that the second integral over τ in (B1), the one from τ_0 to ∞ , turns zero because the relevant Dirac operator annihilates the antiinstanton zero mode $\gamma_\mu \nabla_{\mu,\text{a-in}} \psi_{\text{a-in}}^{(0)} = 0$.

Hence we need to consider in (B1) only the first integral over τ , which runs over the interval $-\infty < \tau < \tau_0$. Integrating over τ by parts and using the fact that the relevant Dirac operator annihilates the fermion zero mode, $\gamma_\mu \nabla_{\mu,\text{in}} \psi_{\text{in}}^{(0)} = 0$, we obtain

$$h_s = i \operatorname{sign}\left(s + \frac{1}{2}\right) \int \left(\psi_{\text{in}}^{(0)}(\mathbf{r}, \tau_0)\right)^+ \gamma_4 \psi_{\text{a-in}}^{(0)}(\mathbf{r}, -\tau_0) d^3 r. \quad (\text{B3})$$

For $s < 0$ the result looks very similar, but the sign in front of the final expression changes. Having this in mind, the factor $\operatorname{sign}(s + \frac{1}{2})$ was incorporated in Eq.(B3) to make it applicable for all $s = 0, \pm 1, \dots$. We see that in accord with our preliminary expectations the integral over τ collapses to the point $\tau = \tau_0$.

Further calculations are straightforward. Using Eqs.

(4.3) and (4.4) one obtains

$$\begin{aligned} & \int \left(\psi_{\text{in}}^{(0)}(\mathbf{r}, \tau_0)\right)^+ \gamma_4 \psi_{\text{a-in}}^{(0)}(\mathbf{r}, -\tau_0) d^3 r = \\ & \frac{1}{\pi^2} \int \operatorname{Tr}(\sigma_\mu^- \sigma_\nu^-) \frac{n_\mu n'_\nu}{(x^2 + 1)^{3/2} (x'^2 + 1)^{3/2}} d^3 r = \\ & - \frac{2}{\pi^2} \int \frac{d^3 r}{(x^2 + 1)^3} = - \frac{1}{2(\tau_0^2 + 1)^{3/2}} \approx - \frac{4\beta^3 \operatorname{sign}(s + \frac{1}{2})}{(s + \frac{1}{2})^3}. \end{aligned} \quad (\text{B4})$$

Here the intermediate expressions incorporate the variables $x_\mu = (\mathbf{r}, \tau_0)$, $n_\mu = x_\mu / \sqrt{x^2}$, as well as $x'_\mu = (\mathbf{r}, -\tau_0)$ and $n'_\mu = x'_\mu / \sqrt{x'^2}$. We find from these conditions that

$$\sigma_\nu^- n'_\nu = -\sigma_\nu^+ n_\nu, \quad (\text{B5})$$

and consequently

$$\operatorname{Tr}(\sigma_\mu^- \sigma_\nu^-) n_\mu n'_\nu = -\operatorname{Tr}(\sigma_\mu^- \sigma_\nu^+) n_\mu n_\nu = -2. \quad (\text{B6})$$

In the third line in (B4) we use the fact that $x^2 = x'^2$. The last expression in (B4) is written remembering (B2) and condition $\beta = 1/T \ll 1$. Finally, from (B3) and (B4) we derive

$$h_s \approx - \frac{4i\beta^3}{(s + \frac{1}{2})^3}, \quad s = 0, \pm 1, \dots \quad (\text{B7})$$

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 - [23] Equation (4.11) specifies our choice of the phase for the operator of inversion, compare Chapter 19 of Ref. [17].
 - [24] Compare Eq.(4.12) of this work with Eq.(10.15) of [16]. The notation is quite different, but careful comparison shows their close resemblance.
 - [25] This result agrees with a preliminary estimate obtained previously with J.Diacoumis.